MIT OCW GR PSET 4

1 . Connection in Rindler Spacetime; the spacetime for an accelerated observer from pset 2 was:

$$ds^2 = -(1+g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$$

compute all non-zero Christoffels for this spacetime. Problem 3.3 from pset 3 should help here:

$$[g_{\nu r}] = \begin{array}{cc} \begin{array}{cccc} t & x & y & z \end{array} & \\ \begin{bmatrix} -(1+g\bar{x})^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} t \\ x \\ y \\ z \end{array} \end{array}$$

$$0 = \Gamma^{\bar{t}}_{\bar{x}\bar{y}} = \Gamma^{\bar{t}}_{\bar{y}\bar{z}} = \Gamma^{\bar{t}}_{\bar{x}\bar{z}} = \Gamma^{\bar{x}}_{\bar{t}\bar{y}} = \Gamma^{\bar{x}}_{\bar{t}\bar{z}} = \Gamma^{\bar{y}}_{\bar{t}\bar{x}} = \Gamma^{\bar{y}}_{\bar{t}\bar{z}} = \Gamma^{\bar{z}}_{\bar{t}\bar{x}} = \Gamma^{\bar{z}}_{\bar{t}\bar{y}}$$

$$\boxed{\Gamma^{\bar{x}}_{\bar{t}\bar{t}}} = -\frac{1}{2}(g_{\bar{x}\bar{x}})^{-1}\partial_{\bar{x}} g_{\bar{t}\bar{t}} = \left(-\frac{1}{2}\right)\partial_{\bar{x}}\left(-(1+g\bar{x})^2\right)$$

$$\boxed{= g(1+g\bar{x})}$$

$$\boxed{\Gamma^{\bar{y}}_{\bar{t}\bar{t}} = \Gamma^{\bar{z}}_{\bar{t}\bar{t}} = 0}$$

$$\boxed{\Gamma^{\bar{t}}_{\bar{x}\bar{x}} = \Gamma^{\bar{t}}_{\bar{y}\bar{y}} = \Gamma^{\bar{t}}_{\bar{z}\bar{z}} = 0}$$

$$\boxed{\Gamma^{\bar{x}}_{\bar{y}\bar{y}} = \Gamma^{\bar{x}}_{\bar{z}\bar{z}} = 0}$$

$$\boxed{\Gamma^{\bar y}_{\bar x \bar x} = \Gamma^{\bar y}_{\bar z \bar z} = \Gamma^{\bar z}_{\bar x \bar x} = \Gamma^{\bar z}_{\bar y \bar y} = 0}$$

$$\boxed{\Gamma^{\bar t}_{\bar t \bar t} = \Gamma^{\bar y}_{\bar y \bar y} = \Gamma^{\bar z}_{\bar z \bar z} = 0}$$

$$\Gamma^{\bar x}_{\bar x \bar x} = \partial_{\bar x} \ln\left(\sqrt{|g_{\bar x \bar x}|}\right) = \partial_{\bar x} \ln\left(\sqrt{1}\right) = \boxed{\emptyset}$$

$$\Gamma^{\bar y}_{\bar y \bar t} = \partial_{\bar t} \ln\left(\sqrt{|g_{\bar y \bar y}|}\right) = 0$$

$$\boxed{\Gamma^{\bar x}_{\bar x \bar t} = \Gamma^{\bar z}_{\bar z \bar t} = \Gamma^{\bar x}_{\bar x \bar y} = \Gamma^{\bar t}_{\bar t \bar y} = \Gamma^{\bar z}_{\bar z \bar y} = \Gamma^{\bar x}_{\bar x \bar z} = \Gamma^{\bar t}_{\bar t \bar z} = \Gamma^{\bar y}_{\bar y \bar z} = \theta}$$

$$\boxed{\Gamma^{\bar y}_{\bar y \bar x} = \Gamma^{\bar z}_{\bar z \bar x} = 0}$$

$$\Gamma^{\bar t}_{\bar t \bar x} = \partial_{\bar x} \ln\left(\sqrt{|g_{\bar t \bar t}|}\right)$$

$$= \partial_{\bar x} \ln(1 + g\bar x)$$

$$\boxed{\Gamma^{\bar t}_{\bar t \bar x} = \frac{g}{1 + g\bar x}}$$

· And every other Christoffel equals $\emptyset$.

The only non-zero Γ's are:

$$\Gamma^{\bar x}_{\bar t \bar t} = g(1 + g\bar x) \quad \text{and} \quad \Gamma^{\bar t}_{\bar t \bar x} = \frac{g}{1 + g\bar x}$$

$\boxed{2}$ $\boxed{a}$ · Starting from the stress energy tensor for a perfect fluid $T^{\alpha\beta} = (\rho + P) u^{\alpha} u^{\beta} + P g^{\alpha\beta}$ and using local energy momentum conservation s.t. $\nabla_{\alpha} T^{\alpha\beta} = 0$; derive the relativistic Euler equation:

$$(\rho + P) \nabla_{\vec{u}} \vec{u} = -\overline{\overline{h}} \cdot \overline{\nabla} P$$

· Given equations:

$$T^{\alpha\beta} = (\rho + P) u^{\alpha} u^{\beta} + P g^{\alpha\beta}$$

$$\nabla_{\alpha} T^{\alpha\beta} = 0$$

$$h^{\alpha\beta} \equiv g^{\alpha\beta} + u^{\alpha} u^{\beta} = \text{"projection operator"}$$

$$\rightarrow h^{\alpha\beta} u_{\beta} = h^{\alpha\beta} u_{\alpha} = h^{\alpha}_{\ \beta} u^{\beta} = h^{\alpha}_{\ \beta} u_{\alpha} = \emptyset$$

· Begin by taking divergence of $\overline{\overline{T}}$:

$$0 = \nabla_{\alpha} T^{\alpha\beta} = u^{\alpha} u^{\beta} \nabla_{\alpha} (\rho + P) + (\rho + P)(u^{\beta} \nabla_{\alpha} u^{\alpha} + u^{\alpha} \nabla_{\alpha} u^{\beta})$$

$$+ g^{\alpha\beta} \nabla_{\alpha} P + P \underbrace{\nabla_{\alpha} g^{\alpha\beta}}_{0}$$

• Now apply h^α_β to both sides:

$$\emptyset = h^\alpha_\beta \nabla_\alpha T^{\alpha\beta} = U^\alpha \overbrace{h^\alpha_\beta U^\beta}^{\to\ \emptyset} \nabla_\alpha (\rho + P)$$

$$+ (\rho + P)\left(\overbrace{h^\alpha_\beta U^\beta \nabla_\alpha U^\alpha}^{\to\ \emptyset} + h^\alpha_\beta U^\alpha \nabla_\alpha U^\beta \right)$$

$$+ h^\alpha_\beta g^{\alpha\beta} \nabla_\alpha P$$

relabel dummy indices $\alpha \to \nu$

$$\to \emptyset = (\rho + P)\left(h^\alpha_\beta U^\nu \nabla_\nu U^\beta \right) + \underbrace{h^{\alpha\nu} \nabla_\nu P}_{\equiv\ \bar{\bar{h}} \cdot \bar{\nabla} P}$$

$$\to \emptyset = \bar{\bar{h}} \cdot \bar{\nabla} P + (\rho + P)\left(g^\alpha_\beta + U^\alpha U_\beta \right)\left(U^\nu \nabla_\nu U^\beta \right)$$

$$\emptyset = \bar{\bar{h}} \cdot \bar{\nabla} P + (\rho + P)\Big(\underbrace{U^\nu \nabla_\nu U^\alpha}_{\equiv\ \nabla_{\vec{u}} \vec{u}} + U^\alpha U_\beta U^\nu \nabla_\nu U^\beta \Big)$$

$$\to \emptyset = \bar{\bar{h}} \cdot \bar{\nabla} P + (\rho + P) \nabla_{\vec{u}} \vec{u} + \textcircled{A} \quad \text{where}$$

$$\textcircled{A} \equiv U^\alpha U_\beta U^\nu \nabla_\nu U^\beta \quad \rightsquigarrow$$

- As an aside, note that:

$$[redacted] \nabla_\gamma (U_\beta U^\beta)$$

$$= U_\beta \nabla_\gamma U^\beta + U^\beta \nabla_\gamma U_\beta$$

$$= U_\beta \nabla_\gamma U^\beta + U_\beta \nabla_\gamma U^\beta \quad \text{flip indices}$$

$$= 2 U_\beta \nabla_\gamma U^\beta \leftarrow \text{which looks like } (\star)$$

- However, $\nabla_\gamma (U_\beta U^\beta) = \nabla_\gamma (-1) = \emptyset$

$$\rightarrow (\star) = \frac{U^\alpha U^\gamma}{2} \nabla_\gamma (U_\beta U^\beta) = \emptyset$$

$\rightarrow$ <u>overall</u>; $\quad \emptyset = \bar{\bar{h}} \cdot \bar{\nabla} P + (\rho + P) \nabla_{\vec{u}} \vec{U}$

$$\rightarrow \boxed{(\rho + P) \nabla_{\vec{u}} \vec{U} = -\bar{\bar{h}} \cdot \bar{\nabla} P}$$

- As we wanted to show $\checkmark\checkmark$

$\boxed{2}$ $\boxed{b}$ · For a non-relativistic fluid ($\rho \gg P$, $u^t \gg u^i$) and a cartesian basis, show that the relativistic equation reduces to:

$$\frac{\partial v_i}{\partial t} + v_i \partial_i v_j = -\frac{1}{\rho} \partial_i P$$

<u>Given:</u>

$$(\rho + P)(u^\alpha \nabla_\alpha u^\beta) = -h^{\alpha\beta} \nabla_\alpha P$$

- Cartesian basis $\rightarrow \Gamma\text{'s} = 0$ and $\nabla_\alpha \rightarrow \partial_\alpha$
- Also apply $\rho \gg P$ to LHS:

$$\rightarrow \rho u^\alpha \partial_\alpha u^\beta \approx -(g^{\alpha\beta} + u^\alpha u^\beta) \partial_\alpha P$$

- WRite out the LHS sum explicitly:

$$\text{LHS} = \rho\big(\overbrace{u^i \partial_i u^t + u^t \partial_t u^t}^{\text{timelike piece}}, u^i \partial_i u^j + u^t \partial_t u^j \big)$$

- For non-relativistic limit $\gamma \rightarrow 1$

$$\vec{u} \equiv (\gamma, \gamma \vec{v}) \rightarrow (\underset{u^t}{\uparrow}\!\!\!\!1, \underset{u^i}{\uparrow}\!\!\!\!\vec{v})$$

$$\Rightarrow \text{LHS} = \rho\left(0,\ u^i \partial_i u^j + \frac{\partial u^j}{\partial t}\right)$$

· Now the RHS:

$$\text{RHS} = -\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\partial_\alpha P$$

· Think of this as a matrix times a vector:

$g^{\alpha\beta} \approx \eta^{\alpha\beta}$ in flat-spacetime:

$$g^{\alpha\beta} \approx \text{diagonal}(-1,1,1,1)$$

$$u^\alpha u^\beta = \begin{bmatrix} (u^t)^2 & u^{ti} \\ u^{it} & u^{ii} \end{bmatrix} \leftarrow 4\times 4 \text{ matrix}$$

· In the limit, $u^i \ll u^t$, implies

$$u^\alpha u^\beta \approx \text{diagonal}(1,0,0,0)$$

$$\rightarrow h^{\alpha\beta} \approx -\text{diagonal}(0,1,1,1)$$

$$\partial_\alpha P = \begin{bmatrix} \partial_t P \\ \partial_x P \\ \partial_y P \\ \partial_z P \end{bmatrix} \rightarrow \text{RHS} \approx (0, -\partial_i P)$$

Timelike component

space like

· So overall;

$$\boxed{\frac{\partial v_i}{\partial t} + v_i \partial_i v_j \approx -\frac{\partial_i P}{\rho}}$$ as wanted to show...

[2] [c] · Apply the relativistic Euler equation to Rindler spacetime for hydrostatic equilibrium. I.e. the fluid is at rest in the $\bar{x}$ coordinates or $U^{\bar{x}} = 0$. Suppose the EOS is $P = w\rho$. Find $\rho(\bar{x})$ given that $\rho(0) = \rho_0$:

· For Rindler spacetime:

$$[g_{\bar{\alpha}\bar{\beta}}] = \operatorname{diag}(-(1+g\bar{x})^2, 1, 1, 1)$$
$$[g^{\bar{\alpha}\bar{\beta}}] = [g_{\bar{\alpha}\bar{\beta}}]^{-1} = \operatorname{diag}\left(-(1+g\bar{x})^{-2}, 1, 1, 1\right)$$

· Relativistic Euler eqn:

$$(\rho + P)\, U^{\bar{\alpha}} \nabla_{\bar{\alpha}} U^{\bar{\beta}} = -h^{\bar{\alpha}\bar{\beta}} \nabla_{\bar{\alpha}} P$$ ← Tensorial equation holds in all reference frames

· Given that $U^{\bar{x}} = U^{\bar{y}} = U^{\bar{z}} = 0$

$$\rightarrow \vec{u} = (U^{\bar{t}}, \vec{0})$$

- Now remember $\vec{u} \cdot \vec{u} = -1$ always

$$\rightarrow \vec{u} \cdot \vec{u} = u_{\bar{\alpha}} u^{\bar{\alpha}} = g_{\bar{\alpha}\bar{\beta}} u^{\bar{\alpha}} u^{\bar{\beta}} = -1$$

$$\rightarrow -1 = g_{\bar{t}\bar{t}} (u^{\bar{t}})^2 \leftarrow \text{since } u^{\bar{i}} = 0$$

$$\rightarrow -1 = -(1 + g\bar{x})^2 (u^{\bar{t}})^2$$

$$\rightarrow u^{\bar{t}} = 1/(1 + g\bar{x}) \leftarrow \text{We will need to use this later}$$

- Now compress down the relativistic Euler equation to just $\bar{\beta} = \bar{x}$ to get a relation for $\partial p / \partial \bar{x}$:

$$(\rho + p) u^{\bar{\alpha}} \nabla_{\bar{\alpha}} u^{\bar{x}} = -h^{\bar{\alpha}\bar{x}} \nabla_{\bar{\alpha}} p$$

$$\text{LHS} = (\rho + p)\left(+u^{\bar{t}} \nabla_{\bar{t}} u^{\bar{x}} + u^{\bar{i}} \nabla_{\bar{i}} u^{\bar{x}}\right)$$

$\searrow$ 0 since $u^{\bar{x}} = u^{\bar{y}} = u^{\bar{z}} = 0$

$$\text{LHS} = +(\rho + p)(u^{\bar{t}} \nabla_{\bar{t}} u^{\bar{x}})$$

- Remember, we aren't working in a flat spacetime necessarily so $\nabla_{\bar{t}} \neq \partial_{\bar{t}}$

$$\nabla_{\bar{t}} u^{\bar{x}} = \underbrace{\partial_{\bar{t}} u^{\bar{x}}}_{0} + \Gamma^{\bar{x}}_{\bar{t}\bar{r}} u^{\bar{r}} \swarrow \text{only } \Gamma^{\bar{x}}_{\bar{t}\bar{t}} u^{\bar{t}} \text{ is } \neq 0$$

$$\rightarrow \nabla_{\bar{t}} u^{\bar{x}} = g(1 + g\bar{x})(1 + g\bar{x})^{-1} = g$$

$$\rightarrow \text{LHS} = +(\rho + P)\, U^{\bar{t}} \nabla_{\bar{t}} U^{\bar{x}}$$

$$= \frac{+(\rho + P)\, g}{(1 + g\bar{x})}$$. And now use EOS $P = \omega \rho$

$$\rightarrow \text{LHS} = \frac{+\rho g (1 + \omega)}{(1 + g\bar{x})}$$

· Now evaluate the RHS:

$$\text{RHS} = -h^{\bar{\alpha}\bar{x}} \nabla_{\bar{\alpha}} P = -h^{\bar{\alpha}\bar{x}} \partial_{\bar{\alpha}} P$$

$$= -\left(g^{\bar{\alpha}\bar{x}} + \underbrace{U^{\bar{\alpha}} U^{\bar{x}}}_{0}\right)\left(\partial_{\bar{\alpha}} P\right)$$

$$= -g^{\bar{x}\bar{x}} \partial_{\bar{x}} P = -\partial_{\bar{x}} P = -\omega \frac{\partial \rho}{\partial \bar{x}}$$

· Since LHS = RHS, implies:

$$\frac{\partial \rho}{\partial \bar{x}} = \frac{-\rho g (1 + \omega)}{\omega (1 + g\bar{x})} \equiv -\rho K / (1 + g\bar{x})$$

· Let $\bar{y} = 1 + g\bar{x} \rightarrow \frac{\partial}{\partial \bar{y}} = \frac{1}{g} \cdot \frac{\partial}{\partial \bar{x}}$

$$\rightarrow g \frac{\partial \rho}{\partial \bar{y}} = \frac{-K\rho}{\bar{y}} \rightarrow \partial_{\bar{y}} \rho(\bar{y}) = -\left(\frac{K}{g}\right)\left(\frac{\rho}{\bar{y}}\right)$$

$$\rightarrow \int \frac{d\rho}{\rho} = -\frac{k}{g}\int \frac{d\bar{y}}{\bar{y}}$$

$$\rightarrow e^{\ln(\rho)} = e^{-\frac{k}{g}\ln(\bar{y})}$$

$$\rightarrow \rho = \bar{y}^{-k/g} = (1+g\bar{x})\,[redacted]^{\,-(1+w)/w}$$

· But we want the I.C. $\rho(\bar{x}=0)=\rho_0$:

$$\rightarrow \boxed{\rho(\bar{x}) = \rho_0 (1+g\bar{x})\,[redacted]\,\hat{}\,\{-(w+1)/w\}}$$

exponent, no "e" term...

[redacted]

$\boxed{d}$ · Suppose now that $w = w_0/(1+g\bar{x})$. Show that $\rho(\bar{x}) = \rho_0 \exp\{-\bar{x}/L\}$. Find L in terms of the system parameters:

$$\text{LHS} = \frac{d\rho}{d\bar{x}} = \frac{d}{d\bar{x}}(w\rho) = \frac{d}{d\bar{x}}\left(\frac{\rho w_0}{1+g\bar{x}}\right)$$

$$= \left(\frac{w_0}{1+g\bar{x}}\right)\frac{d\rho}{dx} - \frac{g w_0 \rho}{(1+g\bar{x})^2} \rightsquigarrow$$

$$\text{RHS} = -\frac{\rho g\left(1+\frac{\omega_0}{1+g\bar{x}}\right)}{(1+g\bar{x})}$$

$$\to \left(\frac{\omega_0}{1+g\bar{x}}\right)\partial_x \rho = \frac{-\rho g}{1+g\bar{x}} - \frac{\omega_0 \cancel{g}\rho}{(\cancel{1}+g\bar{x})^2} + \frac{\omega_0 \cancel{g}\rho}{(\cancel{1}+g\bar{x})^2}$$

$$\to \frac{d\rho}{dx} = \frac{-g}{\omega_0}\rho \to \int \frac{d\rho}{\rho} = \int \frac{-g}{\omega_0}dx$$

$$\to e^{\ln(\rho)} = e^{-gx/\omega_0} \to \boxed{\rho = \exp\{-gx/\omega_0\}}$$

· Apply I.C: $\rho(0) = \rho_0$

$$\to \rho(\bar{x}) = \rho_0 \exp\{-gx/\omega_0\}$$

· Now add back in the c's (speed of light)

$[g] = m/s^2$

$$[P] = [\omega][\rho] \longrightarrow \frac{N}{\cancel{m^2}} = [\omega]\frac{kg\cdot c^2}{m^3} \leftarrow \text{energy density}$$

$$\to \frac{\cancel{kg}\cdot m}{s^2} = [\omega]\frac{\cancel{kg}\cdot c^2}{m} \to [\omega] = \frac{m^2/s^2}{c^2}$$

$$\left[\frac{g}{\omega_0}\right] = \frac{1}{m} = \left[\frac{1}{L}\right] = \frac{m/s^2}{\frac{m^2/s^2}{c^2}} = \frac{c^2}{m} \text{ so we need}$$

that $\frac{g}{\omega_0} \to \frac{g}{\omega_0 c^2}$ $\longrightarrow$

$$\boxed{\rho(x) = \rho_0 \exp\{-gx/w_0 c^2\}} \quad (*)$$

$$= \rho_0 \exp\{-x/L\}$$

- where $\boxed{L \equiv w_0 c^2 / g}$ ✓

[e] · Compare your solution to the density profile of a non-relativistic, plane-parallel, isothermal atmosphere for which $P = \rho K T / \nu$ in a constant gravity field. Use the nonrelativistic Euler equation with a term $-\partial_i \Phi = g_i$ added to the RHS where Φ is Newtonian gravitational potential & $g_i \approx 9.8 \text{ m/s}^2$.

$$\frac{\partial v_i}{\partial t} + v_k \partial_k v_i = -\frac{1}{\rho}\partial_i P + \partial_i \Phi$$

- Assume $v_i = 0$ (Hydrostatic equilibrium)

$$\rightarrow \frac{1}{\rho}\partial_i P = \partial_i \Phi = -g_i$$

$$\rightarrow \left(\frac{1}{\rho}\right)(\partial_i \rho)\left(\frac{KT}{\nu}\right) = -g_i \rightarrow \partial_i \rho = \frac{-\nu g_i}{KT}\rho$$

[redacted] $\rightarrow \boxed{\rho(z) = \rho_0 \exp\{-\nu g_z z / KT\}}$ (☆)

• Question: Why does Hydrostatic equilibrium in Rindler spacetime - where there is no gravity - give such similar results to hydrostatic equilibrium in a gravitational field?

Answer:

. This is ultimately due to the weak Equivalence Principle. I.e., one cannot discern the difference between the effects of gravity + a uniformly accelerated system.

$\boxed{3}$ · Spherical Hydrostatic equilibrium:

- The line element for a spherically symmetric static spacetime is:

$$ds^2 = -e^{2\phi(r)}dt^2 + \left(1 - \frac{2GM(r)}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

where $\phi(r)$ + $M(r)$ are functions of "r". In hydrostatic equilibrium, [redacted] $U^r = U^\theta = U^\varphi = 0$.

- Use the relativistic Euler equation to derive the diff eqn for pressure:

<u>Given</u> $(\rho + P)\, u^\alpha \nabla_\alpha u^\beta = -h^{\alpha\beta}\nabla_\alpha P$

$$[g_{\alpha\beta}] = \begin{bmatrix} -e^{2\phi} & 0 & 0 & 0 \\ 0 & (1-2GM/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

is our metric for this spacetime ...

$$\vec{u}\cdot\vec{u} = -1 = g_{\alpha\beta}u^\alpha u^\beta = g_{tt}(u^t)^2 = -e^{2\phi}(u^t)^2$$

$\rightarrow \vec{u} = \left(e^{-\phi(r)}, \vec{0}\right)$ which is important to know...

- Now look at the following component of the relativistic Euler equation

$$(\rho + P)\, u^\alpha \nabla_\alpha u^r = - h^{\alpha r}\,\partial_\alpha P$$

$$\to +(\rho + P)\, u^t \nabla_t u^r = -(g^{\alpha r} + \cancelto{0}{u^\alpha u^r})\,\partial_\alpha P$$

$$\to +(\rho + P)\, u^t (\cancelto{0}{\partial_t u^r} + \Gamma^r_{t\nu} u^\nu) = -g^{rr}\,\partial_r P$$

$$\to +(\rho + P)\, u^t\, \Gamma^r_{tt}\, u^t = \left(\frac{2GM}{r} - 1\right)\frac{\partial P}{\partial r}$$

· Aside

$$\Gamma^r_{tt} \equiv \left(-\frac{1}{2}\right) g^{rr}\,\partial_r g_{tt}$$

$$= \left(-\frac{1}{2}\right)\left(1 - \frac{2GM}{r}\right)\partial_r\left(-e^{+2\phi(r)}\right)$$

$$= \left(+\frac{1}{2}\right)\left(+2\frac{\partial \phi}{\partial r}\, e^{+2\phi(r)}\right)\left(1 - \frac{2GM}{r}\right)$$

$$= -\left(\frac{2GM}{r} - 1\right) e^{+2\phi}\,\frac{\partial \phi}{\partial r}$$

$$\to -(\rho + P)\cancel{\left(\frac{2GM}{r} - 1\right)}\cancel{e^{+2\phi}}\,\frac{\partial \phi}{\partial r}\,\cancel{e^{-2\phi}} = \cancel{\left(\frac{2GM}{r} - 1\right)}\frac{dP}{dr}$$

$$\to \boxed{\frac{dP}{dr} = -(\rho + P)\frac{\partial \phi}{\partial r}}$$

$\partial_\theta P = \partial_\varphi P = 0$ due to spherical symmetry

$\boxed{4}$. Converting from non-affine to affine parameterization

- Suppose $V^\alpha = \dfrac{dx^\alpha}{d\lambda^*}$ obeys the geodesic equation in the form $\dfrac{DV^\alpha}{d\lambda^*} = K(\lambda^*)\, V^\alpha$ s.t. clearly λ^* is not an affine parameter. Show that $U^\alpha = \dfrac{dx^\alpha}{d\lambda}$ obeys the geodesic equation in the form $\dfrac{DU^\alpha}{d\lambda} = 0$ as long as:

$$\frac{d\lambda}{d\lambda^*} = \exp\left\{ \int K(\lambda^*)\, d\lambda^* \right\}$$

- **Begin:** $$V^\alpha = \frac{dx^\alpha}{d\lambda} \cdot \frac{d\lambda}{d\lambda^*} = U^\alpha \frac{d\lambda}{d\lambda^*}$$

  - Plug back into relation for V^α:

$$\frac{DV^\alpha}{d\lambda^*} = V^\beta \nabla_\beta V^\alpha = \cancel{\frac{d\lambda}{d\lambda^*}}\, U^\beta \nabla_\beta \left(U^\alpha \frac{d\lambda}{d\lambda^*} \right) = K(\lambda^*) \cancel{\frac{d\lambda}{d\lambda^*}}\, U^\alpha$$

$$\rightarrow \left(U^\beta \nabla_\beta \cancel{U^\alpha} \right) \left(\frac{d\lambda}{d\lambda^*} \right) + U^\alpha U^\beta \frac{\partial}{\partial x^\beta} \left(\frac{d\lambda}{d\lambda^*} \right) = K(\lambda^*)\, U^\alpha$$

$\emptyset$

· The first term on the left $\rightarrow 0$ since $\frac{DU^{\alpha}}{d\lambda} = 0$

So then we have:

$$U^{\beta} \frac{\partial}{\partial x^{\beta}}\left(\frac{d\lambda}{d\lambda^*}\right) = k(\lambda^*) \quad \text{and} \quad U^{\beta} \equiv \frac{dx^{\beta}}{d\lambda}$$

$$\rightarrow \frac{\partial}{\partial \lambda}\left(\frac{d\lambda}{d\lambda^*}\right) = k(\lambda^*)$$

$$\rightarrow \partial\left(\frac{d\lambda}{d\lambda^*}\right) = k(\lambda^*)\,\partial\lambda = k(\lambda^*)\frac{d\lambda}{d\lambda^*}\,\partial\lambda^*$$

$$\rightarrow \int \frac{\partial(d\lambda / d\lambda^*)}{d\lambda / d\lambda^*} = \int k(\lambda^*)\,\partial\lambda^*$$

$$\rightarrow e^{\ln(d\lambda / d\lambda^*)} = e^{\int k(\lambda^*)\,d\lambda^*}$$

$$\rightarrow \boxed{\frac{d\lambda}{d\lambda^*} = \exp\left\{\int k(\lambda^*)\,d\lambda^*\right\} \quad \text{W} \quad \text{Q.E.D.}}$$

$\boxed{5}$ • A particle with conserved charge "e" moves with 4-velocity u^α in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_ν.

• The EOM for this particle is:

$$u^\beta \nabla_\beta U_\alpha = e F_{\alpha\beta} u^\beta$$

where $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$

• The spacetime emits a killing vector field ξ^α such that:

$$\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0 \quad \text{and} \quad \mathcal{L}_{\vec{\xi}}\, A_\alpha = 0$$

• Show that the quantity $(U_\alpha + e A_\alpha)\xi^\alpha$ is a constant along the worldline of the particle:

$$\frac{D}{d\tau}\left(\xi^\alpha(U_\alpha + eA_\alpha)\right) \equiv U^\beta \nabla_\beta\left(\xi^\alpha(U_\alpha + eA_\alpha)\right)$$

$$= U_\alpha U^\beta \nabla_\beta \xi^\alpha + \underbrace{\xi^\alpha U^\beta \nabla_\beta U_\alpha}_{\underbrace{e\,\xi^\alpha F_{\alpha\beta} u^\beta}_{e\left(\xi^\alpha \nabla_\alpha A_\beta - \xi^\alpha \cancel{\nabla_\beta} A_\alpha\right) u^\beta}} + eA_\alpha U^\beta \nabla_\beta \xi^\alpha$$

$$+ e\,\xi^\alpha \cancel{U^\beta} \nabla_\beta A_\alpha$$

$$= U_\alpha U^\beta \nabla_\beta \xi^\alpha + e A_\alpha U^\beta \cancel{\nabla_\beta \xi^\alpha} + e \xi^\alpha (\cancel{\nabla_\alpha A_\beta}) U^\beta$$

- Use the fact $\mathcal{L}_{\vec{\xi}} A_\beta = 0$

$\hookrightarrow \xi^\alpha \nabla_\alpha A_\beta + A_\alpha \nabla_\beta \xi^\alpha = 0$

$\rightarrow \xi^\alpha \nabla_\alpha A_\beta = -A_\alpha \nabla_\beta \xi^\alpha$

- So the last two terms cancel ✓✓

$$\rightarrow \frac{D}{d\tau}\left(\xi^\alpha (U_\alpha + e A_\alpha)\right) = U_\alpha U^\beta \nabla_\beta \xi^\alpha$$

flip α's

$$= U^\alpha U^\beta \nabla_\beta \xi_\alpha$$

$$= U^\alpha U^\beta \left(\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta + \underbrace{\nabla_\beta \xi_\alpha - \nabla_\alpha \xi_\beta}_{\text{anti-sym under } \alpha \leftrightarrow \beta \text{ (so goes to } \to 0)} \right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} U^\alpha U^\beta \left(\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta \right)$$

sym. under $\alpha \leftrightarrow \beta$

Killing's equation via the fact that $\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0$

$$\rightarrow \boxed{\frac{D}{d\tau}\left(\xi^\alpha (U_\alpha + e A_\alpha)\right) = 0 \text{ so it is a constant of motion ✓✓ QED}}$$